J. Fluid Mech. (1972), *vol.* 51, *part* 1, *pp.* 119–135
Printed in Great Britain

Examples of steady vortex rings of small cross-section in an ideal fluid

By L. E. FRAENKEL
Department of Applied Mathematics and Theoretical Physics, Cambridge

(Received 6 October 1970)

The existence theory for steady vortex rings of small cross-section is used to derive asymptotic formulae that describe the shape and overall properties of such rings. A certain two-parameter family of rings is studied in detail, to a first approximation; for members of this family, the ratio ω/r (of vorticity to cylindrical radius) falls from a positive maximum at a central point of the core cross-section to a value at the core boundary that can be substantially smaller or even negative. The case of uniform ω/r is considered to a higher order of approximation, and the formulae given for this case appear to be useful for quite substantial cross-sections.

1. Introduction

The existence of steady vortex rings in an inviscid fluid of uniform density has been proved, for the case of small cross-sections, by Maruhn (1957) and by the present writer (1970), who learned of Maruhn's work only after his own had been published. Although Maruhn's paper is only an outline – the reader being referred for proofs and details to earlier work (Maruhn 1934) on a related problem in Newtonian gravitation theory – this outline makes it clear that, as far as the basis of the existence proof is concerned, the present writer merely recovered Maruhn's ideas. However, the details of the two treatments differ and in Maruhn's work are less complete and contain minor errors that are unimportant for questions of existence, but of some importance for quantitative description of the vortex rings in question. In what follows, we refer to the writer's (1970) paper, henceforth denoted by **3**.

The bounded region to which the vorticity is confined will be called the *core* of the vortex ring. In the present paper we adapt the existence theory, in which the distribution over the core streamlines of ω/r (the ratio of vorticity to cylindrical radius) is more or less arbitrary, to derive explicit, although approximate, descriptions of certain steady vortex rings characterized by particular vorticity distributions and by small cross-sections. To make this account largely self-contained, we give in § 2 a shortened version of the formulation adopted in **3**. From this we proceed to show in § 3 how expansions in powers and logarithms of the small cross-section parameter may be computed; this scheme of successive approximations differs from that used in the existence proof although the two have something in common. Section 4 consists of a list of formulae giving, for the general case, first approximations to the propagation velocity, kinetic energy, etc., in terms of the arbitrary function defining the vorticity distribution.

A particular family of vorticity functions is introduced in § 5; it is such that ω/r decreases from a positive maximum at a central point in the core cross-section to a value at the core boundary that may be positive, zero or negative. The effects on overall properties like propagation velocity of varying this vorticity distribution are displayed, and are found to be numerically small for fixed circulation and fixed cross-sectional area as long as the vorticity remains one-signed.

In § 6 we consider rings for which ω/r is constant throughout the core. This simplest of all admissible vorticity distributions has been a favourite for over a century: it characterizes the rings of small cross-section considered by Helmholtz, Kelvin and Hicks (see Lamb 1932, § 163) when that early work is interpreted correctly; it is the vorticity distribution of Hill's spherical vortex, which is the only steady 'ring' represented by an exact solution in closed form; it satisfies the vorticity equation of a viscous fluid (although the corresponding interface condition is violated by our examples); and it is the vorticity of the Prandtl–Batchelor theorem about the inviscid limit of flows with closed streamlines. For our purposes, the most relevant treatment of this case is that of Dyson (1893); I am indebted to Prof. P. G. Saffman for calling my attention to this remarkable paper, which Lamb mentions only in the context of stability and oscillations. Dyson considered a variety of problems related to the gravitational potential of a torus, and, assuming the existence of solutions in all cases, proceeded to calculate approximations of a high order (for small cross-sections) by expanding certain operators in a manner that is as impressive as it is bewildering to modern eyes. In the present paper we recover and enlarge upon Dyson's results for steady vortex rings by applying our general theory, against the background of an existence proof, to the particular case of uniform ω/r. More precisely, we compute the shape of the cross-section to one order less than Dyson, and the propagation velocity to the same order as he did; at the same time, our formulae describe explicitly the flow in the core; and, finally, we make a comparison with Hill's spherical vortex which suggests that these results are useful for quite substantial cross-sections. In fact, Dyson's formula for the propagation velocity, normalized a little differently in § 6, is seen to be in error by less than 6 % for Hill's spherical vortex even though a supposedly small cross-section parameter is equal to $\sqrt{2}$ in that case.

2. Formulation of the problem

Consider the flow of an inviscid fluid of uniform density occupying all space, and assume symmetry about the axis Oz of cylindrical polar co-ordinates (r, θ, z). This co-ordinate frame is to be fixed with respect to a steady vortex ring represented in a meridional plane (θ = constant) by a cross-section $\mathscr{A}$ with boundary $\partial\mathscr{A}$ and area $|\mathscr{A}|$, as shown in figure 1, and by a stream function $\psi(r, z)$. $\mathscr{A}$ is an open set with closure $\bar{\mathscr{A}} \equiv \mathscr{A} + \partial\mathscr{A}$. Thus, writing vectors in terms of their components with respect to (r, θ, z), we seek a steady velocity field

$$\mathbf{v}(r,z) = (-\psi_z/r, 0, \psi_r/r)$$

generated by vorticity curl $\mathbf{v} = (0, \omega, 0)$ in $\mathscr{A}$ and by a uniform stream with velocity $(0, 0, -\tilde{W})$. This latter corresponds to the propagation velocity $(0, 0, \tilde{W})$ of the ring when it moves through fluid at rest at infinity. We write $\omega/r = F(\psi)$ in accordance with the vorticity equation, and, regarding $F(\psi)$ and $|\mathscr{A}|$ as prescribed, seek ψ, $\partial\mathscr{A}$ and $\tilde{W}$ such that

$$\psi(r,z) = -\tfrac{1}{2}\tilde{W}r^2 + \frac{1}{2\pi}\iint_{\mathscr{A}} \tilde{G}(r,\hat{r},z-\hat{z})\,F(\hat{\psi})\,d\hat{r}\,d\hat{z}, \tag{2.1}$$

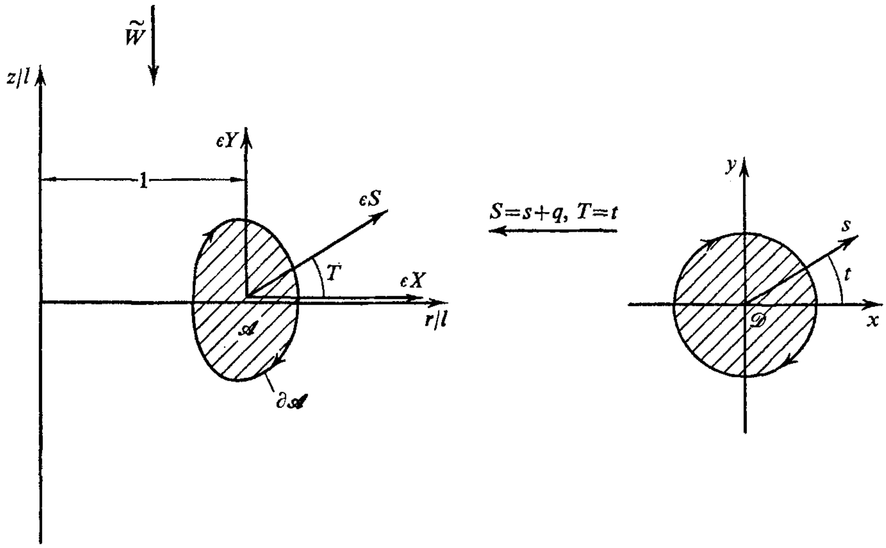


FIGURE 1. Notation. The cross-section $\mathscr{A}$ of a steady vortex ring is the image under the mapping (2.6) of the cross-section $\mathscr{D}$ of a vortex cylinder.

where $\hat{\psi} \equiv \psi(\hat{r},\hat{z})$, ψ is to be constant on $\partial\mathscr{A}$, and

$$\tilde{G} = \tfrac{1}{2}r\hat{r}^2\int_{-\pi}^{\pi}\frac{\cos\theta\,d\theta}{\{r^2+\hat{r}^2-2r\hat{r}\cos\theta+(z-\hat{z})^2\}^{\frac{1}{2}}}. \tag{2.2}$$

The kernel $\tilde{G}$ is the stream function at (r,z) of a singular vortex circle about the axis of symmetry, through the point $(\hat{r},\hat{z})$, and carrying a circulation $2\pi\hat{r}$. Equation (2.1) can be regarded, for points (r,z) on $\bar{\mathscr{A}}$, as a nonlinear integral equation for ψ, $\partial\mathscr{A}$ and $\tilde{W}$; once this equation is solved for points on $\bar{\mathscr{A}}$ the equation defines ψ elsewhere.

The *characteristic radius* l of the vortex ring is defined by writing $r = l$, $z = 0$ for the position of the stagnation point ($\mathbf{v} = \mathbf{0}$) in $\mathscr{A}$; the *cross-section parameter* ϵ is a small positive number appearing in the prescription (2.5) of $F(\psi)$ below, and will turn out to be such that

$$\text{on}\quad \partial\mathscr{A}\colon (r-l)^2+z^2 \sim \epsilon^2 l^2 \quad\text{as}\quad \epsilon\to 0.$$

We introduce non-dimensional co-ordinates (X, Y) and (S, T) by writing

$$r/l-1 = \epsilon X = \epsilon S\cos T,\quad z/l = \epsilon Y = \epsilon S\sin T,$$

and, letting $\mathbf{S}$ denote the co-ordinate pair (S, T), define

$$|\mathbf{S}-\hat{\mathbf{S}}| = \{S^2+\hat{S}^2-2S\hat{S}\cos(T-\hat{T})\}^{\frac{1}{2}}.$$

Now the kernel $\tilde{G}(r,\hat{r},z-\hat{z}) \equiv l^2G(\mathbf{S},\hat{\mathbf{S}},\epsilon)$, say, may be approximated by the plane-flow stream function of a straight vortex line (normal to our meridional plane and through its point $(\hat{r},\hat{z})$):

$$G(\mathbf{S},\hat{\mathbf{S}},\epsilon) = \{\ln(8/\epsilon|\mathbf{S}-\hat{\mathbf{S}}|)-2\}\{1+O(\epsilon S+\epsilon\hat{S})\}. \tag{2.3}$$

This suggests that we consider an analogous plane-flow problem. Let (x, y) be Cartesian, and (s, t) polar co-ordinates in the plane; $x = s\cos t$ and $y = s\sin t$. Let there be vorticity $\Omega(s)$ on the unit disk $\mathscr{D}$: $s < 1$, $-\pi < t \leqslant \pi$. Then the plane-flow stream function of the *vortex cylinder* with cross-section $\mathscr{D}$ has the following alternative forms:

$$\Psi(s) = -\int_0^s \frac{d\sigma}{\sigma}\int_0^\sigma \Omega(\rho)\,\rho\,d\rho, \tag{2.4a}$$

$$= \frac{1}{2\pi}\iint_{\mathscr{D}} \ln\frac{\hat{s}}{|\mathbf{s}-\hat{\mathbf{s}}|}\,\Omega(\hat{s})\,\hat{s}\,d\hat{s}\,d\hat{t}, \tag{2.4b}$$

the definitions of $\mathbf{s}$ and $|\mathbf{s}-\hat{\mathbf{s}}|$ being similar to those of $\mathbf{S}$ and $|\mathbf{S}-\hat{\mathbf{S}}|$. We write $V \equiv \Psi'(s)$ for the velocity, demand that† $V \leqslant -cs$ on $\overline{\mathscr{D}}$ for some constant $c > 0$, and normalize Ω to have a mean value of 1:

$$-V(1) = \int_0^1 \Omega(s)\,s\,ds = \tfrac{1}{2}. \tag{2.4c}$$

Apart from these conditions, and certain smoothness restrictions stated in **3** and amply satisfied in the examples below, $\Omega(s)$ is an arbitrary function. We specify $\Omega = 0$ for $s > 1$, but $\Omega(1)$ need not be zero.

Returning to the axisymmetric problem, we now make equation (2.1) tractable by the following two steps.

(i) *The vorticity function $\omega/r = F(\psi)$ is prescribed parametrically by*

$$\left.\begin{aligned}\omega/r &= (U/\epsilon^2l^2)\,\Omega(s)\\ \psi &= Ul^2\{\Psi(s)+B(\epsilon)\}\end{aligned}\right\}\quad (0 \leqslant s \leqslant 1), \tag{2.5}$$

where U is a reference velocity specifying the intensity of the vorticity, Ω and Ψ are as in (2.4), s is the variable (or 'parameter') of the parametric representation, and the 'constant' $B(\epsilon)$, which is *not* prescribed but will have to be found, is the value of ψ/Ul^2 at the image in the rz plane of the point $s = 0$. This image will turn out to be the stagnation point $(l, 0)$, so that $B(\epsilon)$ measures the flux of fluid between the z axis, on which $\psi = 0$ by (2.1), and the stagnation circle $r = l$, $z = 0$; it will be called the *flux constant*.

(ii) *We seek a mapping*

$$S = s+q(s,t,\epsilon),\quad T = t, \tag{2.6}$$

of the (closed) cross-section $\overline{\mathscr{D}}$ of the vortex cylinder onto the (closed) cross-section $\overline{\mathscr{A}}$ of a steady vortex ring; the function q is to be such that the integral equation (2.1) *is satisfied on $\overline{\mathscr{A}}$*. In order that the corresponding mapping between (X, Y) and

† This condition prevents difficulties with the solution of equation (3.8).

(x,y) be one-to-one, *we demand that q be $O(s)$ for $s \to 0$; also $\nabla q \equiv (q_x, q_y)$ is to be uniformly bounded for* $0 < s \leqslant 1$. (However, ∇q turns out to be many-valued at $x = y = 0$, depending on the value of t as $s \to 0$ along a radial line.)

The stagnation point $s = 0$ in the xy plane is thus mapped onto the stagnation point $S = 0$ $(r = l, z = 0)$ in the rz plane, and the streamlines in $\mathscr{A}$ are described by

$$S = s_0 + q(s_0, T, \epsilon), \quad (s = \text{constant} = s_0),$$

with $s_0 = 1$ on $\partial\mathscr{A}$.

Substitution of (2.5) and (2.6) into (2.1) yields the following equation for the function $q(s,t,\epsilon)$ and the constants $\tilde{W}(\epsilon)$, $B(\epsilon)$:

$$\Psi(s) + B = -\tfrac{1}{2}W\{1+\epsilon(s+q)\cos t\}^2 + \frac{1}{2\pi}\iint_{\mathscr{D}} G(\mathbf{s}+\mathbf{q}, \hat{\mathbf{s}}+\hat{\mathbf{q}}, \epsilon)$$
$$\times \left(1+\frac{\hat{q}}{\hat{s}}\right)(1+\hat{q}_{\hat{s}})\,\hat{\Omega}\hat{s}\,d\hat{s}\,d\hat{t} \quad (0 \leqslant s \leqslant 1), \tag{2.7}$$

where $W = \tilde{W}/U$, $\mathbf{s}+\mathbf{q}$ denotes the pair $(s+q(s,t),t)$, and the convention $\hat{q} \equiv q(\hat{s},\hat{t})$, $\hat{\Omega} \equiv \Omega(\hat{s})$, etc. is adopted throughout the paper.

3. The expansion scheme

It is proved in **3** that (2.7) has a solution $\{q, W, B\}$ with the desired properties for sufficiently small values of ϵ; here we wish to compute approximations to that solution and to the properties of the vortex ring. To this end, we substitute into (2.7) the following expansion of G (Lamb 1932, §161; Fraenkel 1969, §3):

$$G(\mathbf{S}, \hat{\mathbf{S}}, \epsilon) = \ln\frac{8}{\epsilon|\mathbf{S}-\hat{\mathbf{S}}|}\left\{1+\sum_1^\infty \epsilon^n p_n(\mathbf{S},\hat{\mathbf{S}})\right\} + \left\{-2+\sum_1^\infty \epsilon^n P_n(\mathbf{S},\hat{\mathbf{S}})\right\}, \tag{3.1a}$$

where the p_n and P_n are homogeneous polynomials of degree n in $X, \hat{X}$ and $Y-\hat{Y}$, and are even functions of $Y-\hat{Y}$. In particular,

$$\left.\begin{aligned}
p_1 &= \tfrac{1}{2}(X+3\hat{X}), & P_1 &= -\tfrac{1}{2}(X+5\hat{X}),\\
p_2 &= \tfrac{1}{16}\{(X+3\hat{X})^2+3(Y-\hat{Y})^2\}, & P_2 &= \tfrac{1}{16}\{3X^2-6X\hat{X}-5\hat{X}^2 -(Y-\hat{Y})^2\},\\
p_3 &= -\tfrac{1}{32}\{(X-\hat{X})^3+3(X-\hat{X})\times(Y-\hat{Y})^2\}, & P_3 &= \tfrac{1}{48}\{-X^3+6X^2\hat{X}+3X\hat{X}^2+8\hat{X}^3 +(6X+3\hat{X})(Y-\hat{Y})^2\}.
\end{aligned}\right\} \tag{3.1b}$$

The two series in (3.1a) converge uniformly and absolutely if

$$\epsilon(S+\hat{S}) \leqslant 2-\alpha, \quad \epsilon|X| \leqslant 1-\alpha, \quad \epsilon|\hat{X}| \leqslant 1-\alpha \tag{3.1c}$$

for some (arbitrarily small) $\alpha > 0$.

We now split the integrand in (2.7) into three parts: the first corresponds to $q \equiv 0$, the second is linear in q and contains no factor ϵ, and the third consists of the terms of higher order in q and ϵ. Thus, with the notation

$$\Gamma = \Gamma(\mathbf{s},\hat{\mathbf{s}}) = -\ln|\mathbf{s}-\hat{\mathbf{s}}|, \quad p_n = p_n(\mathbf{s},\hat{\mathbf{s}}), \quad P_n = P_n(\mathbf{s},\hat{\mathbf{s}}),$$
$$\hat{J}_1 = \hat{q}/\hat{s}+\hat{q}_{\hat{s}}, \quad \hat{J}_2 = \hat{q}\hat{q}_{\hat{s}}/\hat{s},$$

(where $1+J_1+J_2$ is the Jacobian $\partial(X,Y)/\partial(x,y)$ of the transformation (2.6)) we write

$$G(\mathbf{s}+\mathbf{q},\hat{\mathbf{s}}+\hat{\mathbf{q}},\epsilon)\,(1+\hat{q}/\hat{s})\,(1+\hat{q}_{\hat{s}}) = G(\mathbf{s},\hat{\mathbf{s}},\epsilon)+H_1+H_2, \tag{3.2a}$$

where

$$H_1 = \Gamma_s q+\Gamma_{\hat{s}}\hat{q}+\{\Gamma+\ln(8/\epsilon)-2\}\,(\hat{q}/\hat{s}+\hat{q}_{\hat{s}}), \tag{3.2b}$$

and

$$\begin{aligned} H_2 = {}& \tfrac{1}{2}\{\Gamma_{ss}q^2+2\Gamma_{s\hat{s}}q\hat{q}+\Gamma_{\hat{s}\hat{s}}\hat{q}^2\}+\{\Gamma_s q+\Gamma_{\hat{s}}\hat{q}\}\{\epsilon p_1+\hat{J}_1\}+\{\Gamma+\ln(8/\epsilon)\} \\ & \times\{\epsilon p_1(\mathbf{q},\hat{\mathbf{q}})+\epsilon\hat{J}_1 p_1+\hat{J}_2\}+\{\epsilon P_1(\mathbf{q},\hat{\mathbf{q}})+\epsilon\hat{J}_1 P_1-2\hat{J}_2\} \\ & +O(q^3, q^2\epsilon\ln\epsilon, \text{etc.}). \end{aligned} \tag{3.2c}$$

(The formal expansion of $\ln|\mathbf{s}+\mathbf{q}-\hat{\mathbf{s}}-\hat{\mathbf{q}}|$ in powers of q and $\hat{q}$ is legitimate because although Γ_{ss}, for example, has a non-integrable singularity at $\hat{s}=s$, the sum $\Gamma_{ss}q^2+2\Gamma_{s\hat{s}}q\hat{q}+\Gamma_{\hat{s}\hat{s}}\hat{q}^2$ is strictly bounded: see (7.12) and (7.13) of **3**. In fact, this expansion of the logarithm converges uniformly if $\sup|\nabla q| < 2^{-\frac{1}{2}}$ and if terms of the same degree in q and $\hat{q}$ are always kept together.)

The three terms on the right of (3.2a) are now multiplied by $\hat{\Omega}/2\pi$ and integrated over $\mathscr{D}$. Referring to (2.4b), we find that

$$\frac{1}{2\pi}\iint_{\mathscr{D}} G(\mathbf{s},\hat{\mathbf{s}},\epsilon)\,\hat{\Omega}\hat{s}\,d\hat{s}\,d\hat{t} = \Psi(s)+C_0-g(\mathbf{s},\epsilon), \tag{3.3a}$$

where

$$C_0 = \int_0^1\left(\ln\left(\frac{8}{\epsilon\hat{s}}\right)-2\right)\hat{\Omega}\hat{s}\,d\hat{s} = \tfrac{1}{2}\ln\left(\frac{8}{\epsilon}\right)-1-\Psi(1), \tag{3.3b}$$

and

$$-g(\mathbf{s},\epsilon) = \frac{1}{2\pi}\iint_{\mathscr{D}}\sum_1^\infty \epsilon^n\left\{\ln\left(\frac{8}{\epsilon|\mathbf{s}-\hat{\mathbf{s}}|}\right)p_n+P_n\right\}\hat{\Omega}\hat{s}\,d\hat{s}\,d\hat{t}. \tag{3.3c}$$

Here the basic stream function Ψ and the known constant C_0 form the dominant part; the known function g is $O(\epsilon\ln\epsilon)$.

For the integral of H_1 we note, again from (2.4b), that $\Gamma_s q$ contributes $\Psi' q \equiv Vq$; we integrate the term $\Gamma_{\hat{s}}\hat{q}$ by parts, obtaining a cancellation with the third term of H_1. Then

$$\frac{1}{2\pi}\iint_{\mathscr{D}} H_1\hat{\Omega}\hat{s}\,d\hat{s}\,d\hat{t} = Vq-\mathbf{L}q+C_1(q), \tag{3.4a}$$

where

$$\mathbf{L}q = \frac{1}{2\pi}\left\{\iint_{\mathscr{D}}\ln\frac{1}{|\mathbf{s}-\hat{\mathbf{s}}|}\,\hat{q}\,\hat{\Omega}'\hat{s}\,d\hat{s}\,d\hat{t}-\Omega(1)\oint_{\hat{s}=1}\ln\frac{1}{|\mathbf{s}-\hat{\mathbf{s}}|}\,\hat{q}\,d\hat{t}\right\}, \tag{3.4b}$$

and

$$C_1(q) = \frac{1}{2\pi}\iint_{\mathscr{D}}\left(\ln\left(\frac{8}{\epsilon}\right)-2\right)\left(\frac{\hat{q}}{\hat{s}}+\hat{q}_{\hat{s}}\right)\hat{\Omega}\hat{s}\,d\hat{s}\,d\hat{t}. \tag{3.4c}$$

The linear integral operator $\mathbf{L}$ is basic in all that follows; the constant C_1 is relatively unimportant. (The use of bold type for both operators and vectors should cause no confusion.)

Finally, we define a nonlinear integral operator $\mathbf{M}$ by

$$\mathbf{M}(q,\epsilon) = \frac{1}{2\pi}\iint_{\mathscr{D}} H_2\hat{\Omega}\hat{s}\,d\hat{s}\,d\hat{t}; \tag{3.5}$$

then (2.7) becomes

$$B = -\tfrac{1}{2}W\{1+\epsilon(s+q)\cos t\}^2+\{C_0-g(\mathbf{s},\epsilon)\}+\{Vq-\mathbf{L}q+C_1(q)\}+\mathbf{M}(q,\epsilon). \tag{3.6}$$

Knowing from (3.3*c*) that

$$g(\mathbf{s},\epsilon) = \sum_1^\infty \epsilon^n g_n(\mathbf{s},\ln\epsilon),$$

we now introduce the similar expansions

$$q(\mathbf{s},\epsilon) = \sum_1^\infty \epsilon^n q_n(\mathbf{s},\ln\epsilon), \quad W = \sum_0^\infty \epsilon^n W_n(\ln\epsilon), \quad B = \sum_0^\infty \epsilon^n B_n(\ln\epsilon),$$

where $g_n, \ldots, B_n$ are polynomials in $\ln\epsilon$; correspondingly,

$$\mathbf{M}(q,\epsilon) = \epsilon^2\mathbf{M}_2(q_1,\ln\epsilon) + \epsilon^3\mathbf{M}_3(q_1,q_2,\ln\epsilon) + \ldots.$$

To save writing, we use the result (proved in appendix B of **3**) that *the q_n are odd or even functions of x ($= s\cos t$) according as n is odd or even, and that*

$$W_{2m+1} = B_{2m+1} = 0.$$

It follows from (3.4*c*) that $C_1(q_{2m+1}) = 0$. Then expansion of (3.6) up to ϵ^3 yields the following equations:

$$B_0 + \tfrac{1}{2}W_0 = C_0, \tag{3.7a}$$

$$Vq_1 - \mathbf{L}q_1 = g_1 + W_0 s\cos t, \tag{3.7b}$$

$$Vq_2 - \mathbf{L}q_2 = \{g_2 + W_0(q_1\cos t + \tfrac{1}{2}s^2\cos^2 t) - \mathbf{M}_2(q_1,\ln\epsilon)\} + B_2 + \tfrac{1}{2}W_2 - C_1(q_2), \tag{3.7c}$$

$$Vq_3 - \mathbf{L}q_3 = \{g_3 + W_0(q_2\cos t + sq_1\cos^2 t) - \mathbf{M}_3(q_1,q_2,\ln\epsilon)\} + W_2 s\cos t. \tag{3.7d}$$

It is useful to note certain general features of the equations (3.7*b, c, d*) before proceeding to particular examples. They are all of the form

$$Vq_n - \mathbf{L}q_n = f(\mathbf{s}) + \alpha + \beta x, \tag{3.8}$$

where f is a known function; we wish to find q_n and either the constant α (n even, $\beta = 0$) or the constant β (n odd, $\alpha = 0$). The label $(\ )_n$ is omitted from f, α and β for ease of writing. We can eliminate α and β from (3.8) by operating on the equation with $(\mathring{\ })$, defined for any smooth function $\phi(\mathbf{s})$ by

$$\mathring{\phi}(\mathbf{s}) = \phi(\mathbf{s}) - \phi(\mathbf{0}) - \{x\phi_x(\mathbf{0}) + y\phi_y(\mathbf{0})\}. \tag{3.9}$$

(Actually, only even functions of y appear in (3.7), so that $\phi_y(\mathbf{0}) = 0$ there. On the other hand, with q_x many-valued at $s = 0$, it is not obvious that $f_x(\mathbf{0})$ is a uniquely defined number for the functions f in (3.7); but this is proved in appendix B of **3**. What happens is that in (3.7*d*), for example, the terms of $(W_0 q_2\cos t)_x$ that are many-valued at $s = 0$ are precisely cancelled by those of $\{\mathbf{M}_3(q_1,q_2,\ln\epsilon)\}_x$.) Since Vq_n is to be $O(s^2)$ for $s \to 0$, application of $(\mathring{\ })$ to (3.8) yields

$$Vq_n - \mathring{\mathbf{L}}q_n = \mathring{f}, \tag{3.10a}$$

where $\mathring{\mathbf{L}}$ is the operator $\mathbf{L}$ with the kernel $\Gamma = -\ln|\mathbf{s}-\hat{\mathbf{s}}|$ replaced by

$$\mathring{\Gamma}(\mathbf{s},\hat{\mathbf{s}}) \equiv \Gamma(\mathbf{s},\hat{\mathbf{s}}) - \Gamma(\mathbf{0},\hat{\mathbf{s}}) - \{x\Gamma_x(\mathbf{0},\hat{\mathbf{s}}) + y\Gamma_y(\mathbf{0},\hat{\mathbf{s}})\}. \tag{3.10b}$$

The linear integral equation (3.10*a*) can be solved more or less explicitly, as is shown in appendix A of **3** and, for particular cases, in what follows. Then, with q_n known, (3.8) determines α and β.

Alternatively, β can be determined directly from (3.8), without computation of q_n, by the following method, which is useful because in the case of (3.7*b*) it gives a formula for the leading term W_0 of the propagation speed. Using the definition (3.4*b*) of $\mathbf{L}$ and the Fourier series

$$\Gamma(\mathbf{s},\hat{\mathbf{s}}) \equiv \ln\frac{1}{|\mathbf{s}-\hat{\mathbf{s}}|} = \begin{cases} \ln\dfrac{1}{s}+\sum_1^\infty \dfrac{1}{n}\left(\dfrac{\hat{s}}{s}\right)^n \cos n(t-\hat{t}) & (s>\hat{s}), \\ \ln\dfrac{1}{\hat{s}}+\sum_1^\infty \dfrac{1}{n}\left(\dfrac{s}{\hat{s}}\right)^n \cos n(t-\hat{t}) & (s<\hat{s}), \end{cases} \tag{3.11}$$

we find that
$$\mathbf{L}\cos t = V(s)\cos t; \tag{3.12}$$

in other words, the homogeneous form of (3.8) has the non-trivial solution $\cos t$. (If γ is a small constant, the corresponding mapping $S = s+\gamma\cos t$, $T = t$ represents a pure translation of the circular streamlines s = constant, to the first order in γ/s.) Let us now recall that $\Omega = 0$ for $s > 1$, and write (3.4*b*) in Stieltjes-integral notation:

$$\mathbf{L}q = \frac{1}{2\pi}\iint \ln\frac{1}{|\mathbf{s}-\hat{\mathbf{s}}|}\,\hat{q}\hat{s}\,d\hat{\Omega}\,d\hat{t}.$$

We define a corresponding symbolic product for any two functions $p(\mathbf{s})$, $q(\mathbf{s})$ sufficiently smooth on $\overline{\mathscr{D}}$:

$$[p,q] = \frac{1}{2\pi}\iint pq\,s\,d\Omega\,dt.$$

Then
$$[p,\mathbf{L}q] = \frac{1}{4\pi^2}\iiiint p(\mathbf{s})\ln\frac{1}{|\mathbf{s}-\hat{\mathbf{s}}|}q(\hat{\mathbf{s}})\,\hat{s}\,d\hat{\Omega}\,d\hat{t}\,s\,d\Omega\,dt = [\mathbf{L}p,q]. \tag{3.13}$$

Using (3.12), (3.13) and (3.8) successively, we obtain

$$0 = [q_n, V\cos t-\mathbf{L}\cos t] = [Vq_n-\mathbf{L}q_n, \cos t] = [f+\alpha+\beta x, \cos t].$$

Now $[\alpha,\cos t] = 0$ by the anti-symmetry of $\cos t$, and integration by parts shows that $[x,\cos t] = -\frac{1}{2}$ under our normalization (2.4*c*). Hence

$$\beta = 2[f,\cos t]. \tag{3.14}$$

Since (3.8) implies (3.14) we can now compute W_0 from (3.7*b*), once g_1 has been evaluated. From the definition (3.3*c*) of g, and the identity $(sV)' = -s\Omega$, we have

$$\begin{aligned} -g_1 &= \frac{1}{2\pi}\iint_{\mathscr{D}}\left\{\ln\left(\frac{8}{\epsilon|\mathbf{s}-\hat{\mathbf{s}}|}\right)(\tfrac{1}{2}s\cos t+\tfrac{3}{2}\hat{s}\cos\hat{t})-(\tfrac{1}{2}s\cos t+\tfrac{5}{2}\hat{s}\cos\hat{t})\right\}(-\hat{s}\hat{V})'\,d\hat{s}\,d\hat{t} \\ &= \tfrac{1}{4}s\cos t\left\{\ln\left(\frac{8}{\epsilon}\right)+\frac{1}{2}+\frac{6}{s^2}\int_0^s \hat{V}\hat{s}^2\,d\hat{s}-2\int_s^1 \hat{V}\,d\hat{s}\right\}, \end{aligned} \tag{3.15}$$

upon using the series (3.11) for the logarithm, integration by parts and the normalization condition. Then by (3.14) we deduce from (3.7*b*) that

$$\begin{aligned} W_0 &= \frac{1}{\pi}\iint g_1(\mathbf{s},\ln\epsilon)\cos t\,s\,d\Omega\,dt \\ &= \frac{1}{4}\left\{\ln(8/\epsilon)-\tfrac{1}{2}+4\int_0^1 V^2(s)\,s\,ds\right\}, \end{aligned} \tag{3.16}$$

and, since $W_1 = 0$, we have $W = W_0\{1+O(\epsilon^2\ln^2\epsilon)\}$.

The result (3.16), which also appears in **3**, has been found independently by Saffman (1970), using heuristic arguments and an energy method that is an extension of Lamb's (1932). It is not clear from this method that the error is *two* orders higher. Saffman goes on to consider vortex rings in a viscous fluid. Maruhn (1957) obtained only the term $\frac{1}{4}\ln(8/\epsilon)$ and underestimated the corresponding error.

4. First approximations to the overall properties of steady vortex rings

In this section we list the properties of principal interest for steady vortex rings, first writing exact definitions (or equivalent expressions given by Lamb 1932, § 162) and then approximate values implied by the present theory. The kinetic energy $\tilde{T}$ is that of the entire fluid when it is at rest at infinity and the ring moves through it; the stream function is then $\psi + \frac{1}{2}\tilde{W}r^2$, where ψ retains the meaning given to it in § 2. The calculations are straightforward and are indicated at the end of the section.

The results are expressed in terms of the characteristic radius l, the cross-section parameter ϵ, the reference velocity U, the non-dimensional velocity $V = \Psi'(s)$ in the vortex cylinder (all defined in § 2) and the density ρ of the fluid. The error factor in all cases is $1 + O(\delta^2)$, where

$$\delta = \epsilon \ln(8/\epsilon),$$

because the first-order displacement ϵq_1 of the streamlines is an odd function of x and because the corresponding perturbations ϵW_1 and ϵB_1, of the propagation velocity and flux constant, are zero.

Cross-sectional area:

$$|\mathscr{A}| = \iint_{\mathscr{A}} dr\,dz = \pi\epsilon^2 l^2\{1 + O(\delta^2)\}. \tag{4.1}$$

Circulation:

$$\kappa \equiv \iint_{\mathscr{A}} \omega\,dr\,dz = Ul\pi\{1 + O(\delta^2)\}. \tag{4.2}$$

Propagation speed:

$$\tilde{W} = \tfrac{1}{4}U\left\{\ln(8/\epsilon) - \tfrac{1}{2} + 4\int_0^1 V^2(s)\,s\,ds\right\}\{1 + O(\delta^2)\}. \tag{4.3}$$

Flux:

$$\psi(l,0) \equiv Ul^2 B = Ul^2\left\{\tfrac{3}{8}\ln(8/\epsilon) - \tfrac{15}{16} - \Psi(1) - \tfrac{1}{2}\int_0^1 V^2(s)\,s\,ds\right\}\{1 + O(\delta^2)\}. \tag{4.4}$$

Impulse:

$$P \equiv \pi\rho \iint_{\mathscr{A}} \omega r^2\,dr\,dz = \rho U l^3 \pi^2\{1 + O(\delta^2)\}. \tag{4.5}$$

Kinetic energy:

$$\begin{aligned}\tilde{T} &\equiv \pi\rho \iint_{\mathscr{A}} (\psi + \tfrac{1}{2}\tilde{W}r^2)\,\omega\,dr\,dz \\ &= \rho U^2 l^3 \pi^2\left\{\tfrac{1}{2}\ln(8/\epsilon) - 1 + 2\int_0^1 V^2(s)\,s\,ds\right\}\{1 + O(\delta^2)\}.\end{aligned} \tag{4.6}$$

These results are found as follows. The expression for $\tilde{W}$ is essentially (3.16). For the kinetic energy we have

$$\frac{\tilde{T}}{\pi\rho U^2 l^3} = \iint_{\mathscr{D}} \{\Psi(s) + B + \tfrac{1}{2}W[1+\epsilon(s+q)\cos t]^2\} \times \Omega(s)\{1+\epsilon(s+q)\cos t\}(1+q/s)(1+q_s)\, s\, ds\, dt,$$

where $q \sim \epsilon q_1$. Hence all terms proportional to δ or ϵ in the integrand are odd functions of x and make no contribution; also $B_1 = W_1 = 0$. It is proved in appendix B of **3** that the ϵ^2 terms have at most a factor $\{\ln(8/\epsilon)\}^3$. Therefore

$$\frac{\tilde{T}}{\pi\rho U^2 l^3} = 2\pi\int_0^1 \{\Psi(s) + B_0 + \tfrac{1}{2}W_0\}\,\Omega(s)\, s\, ds \times \{1+O(\delta^2)\},$$

where $\qquad B_0 + \tfrac{1}{2}W_0 = C_0 = \tfrac{1}{2}\ln(8/\epsilon) - 1 - \Psi(1) \quad$ by (3.7*a*) and (3.3*b*),

$$\int_0^1 \Omega s\, ds = \tfrac{1}{2} \quad \text{by (2.4}c\text{)},$$

$$\int_0^1 \Psi\Omega s\, ds = -\int_0^1 \Psi(sV)'\, ds = \tfrac{1}{2}\Psi(1) + \int_0^1 V^2 s\, ds.$$

This gives (4.6), and the remaining results are found similarly.

5. The vorticity distribution $\omega/r = \text{constant} \times J_0(ks)$

Consider the vortex cylinder described by

$$\left.\begin{aligned} \Omega &= CkJ_0(ks), \quad C = \{2J_1(k)\}^{-1}, \\ V &= -CJ_1(ks), \quad \Psi = (C/k)\{J_0(ks)-1\}, \end{aligned}\right\} \tag{5.1}$$

where J_n is the Bessel function of the first kind of order n, k is a non-negative constant and the limiting values (6.1) are to be understood for $k = 0$. The functions Ω and V are shown in figure 2. Let $j_0 \approx 2{\cdot}40$ and $j_1 \approx 3{\cdot}83$ denote the first positive zeros of J_0 and J_1 respectively; then our theory is applicable for $0 \leqslant k < j_1$. However, it is probably of interest only for $0 \leqslant k \leqslant j_0$, since for $k > j_0$ the vorticity changes sign across some core streamline $s = \text{constant} < 1$.

Corresponding to (5.1) there is a family of steady vortex rings characterized by the two parameters k and ϵ. Their overall properties are given by (4.1) to (4.6), in which

$$\int_0^1 V^2(s)\, s\, ds = \frac{1}{8}\left\{1 - \frac{2}{k}\frac{J_0(k)}{J_1(k)} + \frac{J_0^2(k)}{J_1^2(k)}\right\}. \tag{5.2}$$

As we make the fairly extreme change from $k = 0$ to $k = j_0$ (that is, from the case of uniform ω/r to a vorticity function ω/r that has a maximum at the stagnation point $r = l$, $z = 0$ and is zero on $\partial\mathscr{A}$) the function (5.2) increases from $\frac{1}{16}$ to $\frac{1}{8}$. For $\epsilon = 0{\cdot}2$ and fixed U, l this represents an increase of about 7% in propagation speed and 13% in kinetic energy; for smaller ϵ, the changes are less. Thus the effect on the overall properties of the ring of varying the vorticity distribution appears to be numerically rather small for fixed circulation and one-signed vorticity.

We now determine the function $\epsilon q_1(\mathbf{s}, \ln \epsilon)$, which gives the first correction to the basic pattern of concentric circular streamlines in $\mathscr{A}$. Using (3.15) for g_1, and then applying to (3.7*b*) the operator ($\mathring{}$) defined by (3.9), we obtain

$$Vq_1 - \mathring{\mathbf{L}}q_1 = \mathring{g}_1 \equiv h(s)\cos t, \tag{5.3}$$

where
$$h(s) = (C/k^2)\,[2ks\{1-J_0(ks)\} - \tfrac{3}{2}\{ks - 2J_1(ks)\}]. \tag{5.4}$$

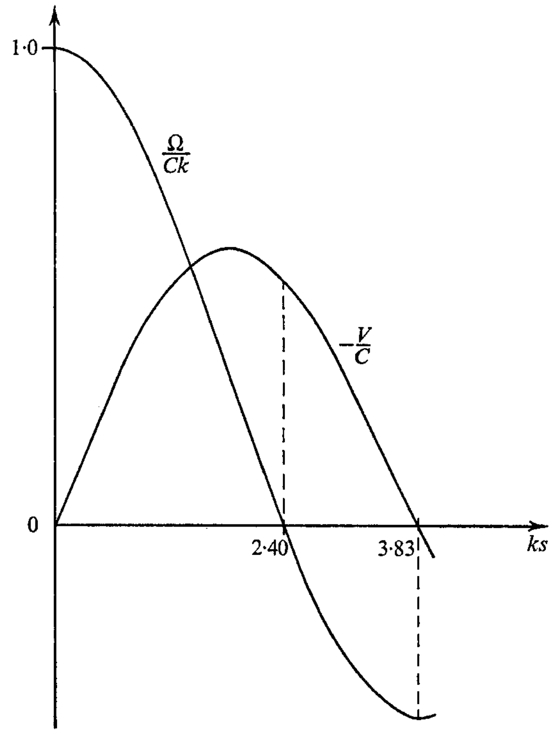


FIGURE 2. Vorticity and velocity functions of the family of vortex cylinders considered in § 5.

We seek the (unique) solution of (5.3) in the form

$$Vq_1 = \{h(s) + w(s)\}\cos t. \tag{5.5}$$

Upon insertion of the Fourier series of $\mathring{\Gamma}(\mathbf{s}, \hat{\mathbf{s}})$ implied by (3.10*b*) and (3.11), equation (5.3) becomes

$$w(s) - \frac{1}{2}\int_0^s \left(\frac{\hat{s}}{s} - \frac{s}{\hat{s}}\right)\frac{\hat{h}+\hat{w}}{\hat{V}}\hat{\Omega}'\hat{s}\,d\hat{s} = 0, \quad (\Omega'/V = k^2).$$

Operating on this with
$$\Delta_1 \equiv \frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \frac{1}{s^2},$$

(this operator is related to the two-dimensional Laplace operator Δ by

$$\Delta\{\phi(s)\cos t\} = \cos t\,\Delta_1\phi),$$

we obtain the ordinary differential equation

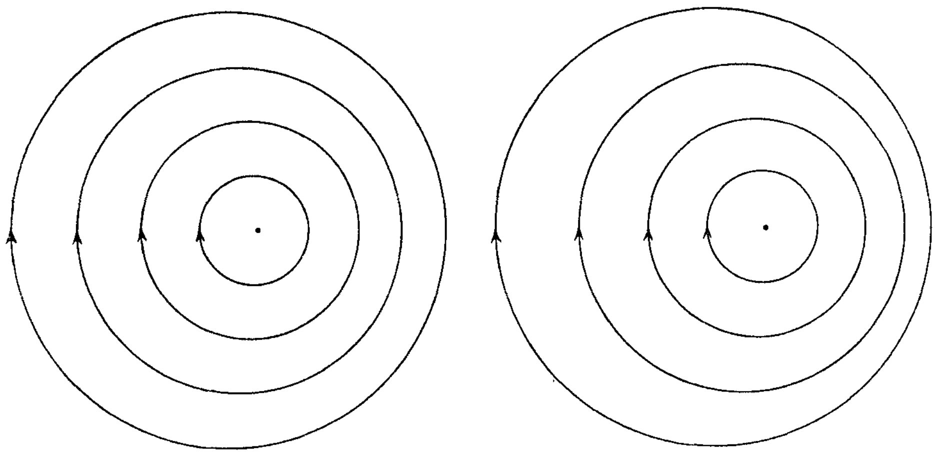
$$(\Delta_1 + k^2)w = -k^2h(s) \quad (0 \leqslant s \leqslant 1). \tag{5.6 a}$$

Also,
$$w(0) = w'(0) = 0, \tag{5.6 b}$$

because Vq_1 is to be $O(s^2)$ for $s \to 0$. Since the solutions $J_1(ks)$, $Y_1(ks)$ of the homogeneous form of (5.6a) are known, the problem (5.6a, b) is an elementary one; the solution is

$$w(s) = (C/k^2)\,[2\{ks - 2J_1(ks) - \tfrac{1}{4}k^2s^2J_1(ks)\} - \tfrac{3}{2}\{ks + ksJ_0(ks) - 4J_1(ks)\}], \quad (5.7)$$

and the desired function q_1 is now given by (5.5), (5.4) and (5.7).



(a) (b)

FIGURE 3. The streamline pattern in the core, according to the approximation

$$S = s + \epsilon q_1(s, T),$$

when $\omega/r = \text{constant} \times J_0(ks)$. The dot marks the stagnation point $S = s = 0$, and the streamlines are $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The axes are oriented as in figure 1. The parameter $\epsilon = 0{\cdot}2$, and $k = 0$, $2{\cdot}40$ for (a), (b) respectively.

The corresponding streamline pattern in the core is shown in figure 3 for $k = 0$, $2{\cdot}40$ and $\epsilon = 0{\cdot}2$. The value $\epsilon = 0{\cdot}2$, which may be too large for neglect of higher order terms when $k = 2{\cdot}40$, has been chosen to make the departure from concentric circles easily visible. For $k = 0$, the distances $|\delta\mathbf{r}|$ between adjacent streamlines in figure 3(a) indicate that the velocity $|\mathbf{v}|$ is still nearly constant (independent of the angle T) on each streamline. For we know that in any axisymmetric flow $|\mathbf{v}|\,r|\delta\mathbf{r}|$ is constant along a streamline pair, to the first order in $|\delta\mathbf{r}|$, and in figure 3(a) the values of $r|\delta\mathbf{r}|$ are nearly constant along each pair. For $k = 2{\cdot}40$, however, the change in $|\delta\mathbf{r}|$ as we move around the core boundary is much more pronounced. For an explanation of this let us start with the pattern for $k = 0$ ($\Omega = 1$) and gradually change the vorticity function Ω to that shown in figure 2 for $k = 2{\cdot}40$. Then the vorticity becomes strong near the stagnation point and weak near the core boundary. On the streamline $s = 1$, say, the velocity is now significantly greater at $T = 0$ than it is at $T = \pi$ because the former point is closer to the concentration of vorticity near the stagnation point. Thus the streamlines moves till closer together near $s = 1$, $T = 0$ and still further apart near $s = 1$, $T = \pi$.

6. The case of uniform ω/r

This case is of special interest, for reasons already mentioned in § 1, and we compute up to ϵ^3. The relevant vortex cylinder is described by

$$\Omega = 1, \quad V = -\tfrac{1}{2}s, \quad \Psi = -\tfrac{1}{4}s^2. \tag{6.1}$$

A simplifying feature is that in the definition (3.4*b*) of the integral operator $\mathbf{L}$ the integral over the disk $\mathscr{D}$ vanishes; the remaining boundary integral is easily evaluated in terms of the Fourier coefficients of $q(1,t,\epsilon)$. This simplification corresponds to the fact that ω/r = constant is a trivial solution of the vorticity equation.

Because of the known form of the forcing function $g(\mathbf{s},\epsilon)$ defined by (3.3*c*), we make explicit the dependence on t of the q_n introduced in § 3, and write

$$q(s,t,\epsilon) = \epsilon q_{1,1}(s)\cos t + \epsilon^2\{q_{2,0}(s) + q_{2,2}(s)\cos 2t\} + \epsilon^3\{q_{3,1}(s)\cos t + q_{3,3}(s)\cos 3t\} + \dots, \tag{6.2}$$

where the polynomial dependence of the $q_{n,m}$ on $\ln\epsilon$ is to be understood. Equations (3.7*a*) and (3.7*b*) of § 3 become, when the latter is divided by $s\cos t$,

$$B_0 + \tfrac{1}{2}W_0 = \tfrac{1}{2}\ln(8/\epsilon) - \tfrac{3}{4}, \tag{6.3}$$

$$-\tfrac{1}{2}q_{1,1}(s) + \tfrac{1}{2}q_{1,1}(1) = -\tfrac{1}{4}\{\ln(8/\epsilon) + 1 - \tfrac{5}{4}s^2\} + W_0. \tag{6.4}$$

Setting $s = 1$ in (6.4) yields W_0, setting $s = 0$ then yields $q_{1,1}(1)$, and the equation now determines $q_{1,1}(s)$. (Of course, (3.16) also gives W_0.) The treatment of (3.7*c*) and (3.7*d*) is similar, the two Fourier components of each equation being treated separately. Some computational details – and checks, for the arithmetic is formidable – are given in the appendix. We find that

$$\left.\begin{aligned} q_1 &= -\tfrac{5}{8}s^2\cos t,\\ q_2 &= \tfrac{97}{256}s^3 + \{[-\tfrac{3}{8}\ln(8/\epsilon) + \tfrac{15}{64}]\,s + \tfrac{101}{256}s^3\}\cos 2t,\\ q_3 &= \{[\tfrac{3}{8}\ln(8/\epsilon) - \tfrac{15}{64}]\,s^2 - \tfrac{7}{8}s^4\}\cos t + \{[\tfrac{51}{128}\ln(8/\epsilon) - \tfrac{291}{1024}]\,s^2 - \tfrac{161}{512}s^4\}\cos 3t.\end{aligned}\right\} \tag{6.5}$$

The overall properties of the vortex ring, defined in § 4 and computed here by the same method as before, are

$$|\mathscr{A}| = \pi\epsilon^2 l^2\{1 + \tfrac{61}{64}\epsilon^2 + O(\delta^4)\}, \tag{6.6}$$

$$\kappa = Ul\pi\{1 + \tfrac{21}{64}\epsilon^2 + O(\delta^4)\}, \tag{6.7}$$

$$\tilde{W} = U\{[\tfrac{1}{4}\ln(8/\epsilon) - \tfrac{1}{16}] + \epsilon^2[\tfrac{37}{256}\ln(8/\epsilon) - \tfrac{223}{1024}] + O(\delta^4)\}, \tag{6.8}$$

$$\psi(l,0) = Ul^2\{[\tfrac{3}{8}\ln(8/\epsilon) - \tfrac{23}{32}] + \epsilon^2[\tfrac{15}{512}\ln(8/\epsilon) - \tfrac{377}{2048}] + O(\delta^4)\}, \tag{6.9}$$

$$P = \rho U l^3\pi^2\{1 - \tfrac{11}{64}\epsilon^2 + O(\delta^4)\}, \tag{6.10}$$

$$\tilde{T} = \rho U^2 l^3\pi^2\{[\tfrac{1}{2}\ln(8/\epsilon) - \tfrac{7}{8}] + \epsilon^2[\tfrac{13}{64}\ln(8/\epsilon) - \tfrac{37}{64}] + O(\delta^4)\}. \tag{6.11}$$

There are certain advantages in rewriting these results in terms of parameters l_* and ϵ_*, defined to be such that $r = l_*$, $z = 0$ is the mid point of the chord $z = 0$ of $\mathscr{A}$, while ϵ_* is an exact measure of the cross-sectional area $|\mathscr{A}|$. Thus we define

$$l_* = \tfrac{1}{2}\{r|_{s=1,\,t=0} + r|_{s=1,\,t=\pi}\}, \tag{6.12a}$$

$$\epsilon_*^2 = |\mathscr{A}|/\pi l_*^2 \quad (\epsilon_* > 0); \tag{6.12b}$$

and, correspondingly, a new reference velocity U_* is defined by

$$\omega/r = U_*/\epsilon_*^2 l_*^2. \tag{6.12c}$$

These new parameters are related more simply to the circulation κ (as well as to the cross-sectional area $|\mathscr{A}|$); the equation of $\partial\mathscr{A}$ is simplified; and ϵ_* is known for Hill's spherical vortex, whereas ϵ is not. From (6.5) and (6.6) we have

$$l_*/l = 1-\tfrac{5}{8}\epsilon^2+O(\delta^4), \quad \epsilon_*/\epsilon = 1+\tfrac{141}{128}\epsilon^2+O(\delta^4), \quad U_*/U = 1+\tfrac{61}{64}\epsilon^2+O(\delta^4), \tag{6.13}$$

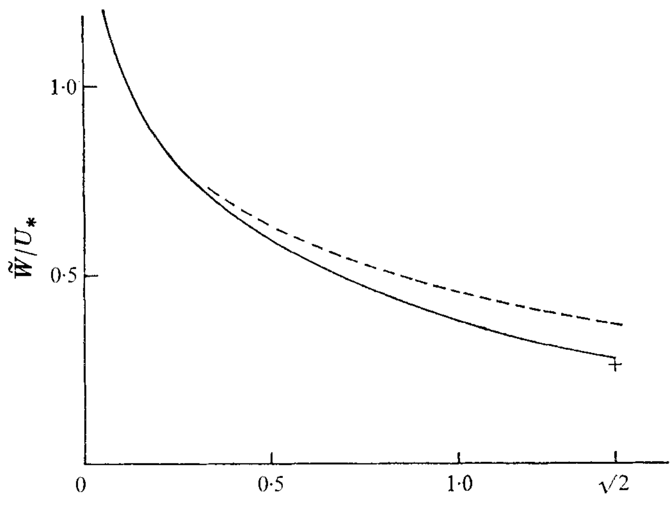


FIGURE 4. Variation of the propagation speed $\tilde{W}$ with the parameter ϵ_* for ω/r = constant. – – – –, $\frac{1}{4}\{\ln(8/\epsilon_*)-\frac{1}{4}\}$; ——, formula (6.16); +, exact result for Hill's vortex.

and so, with $\delta_* \equiv \epsilon_* \ln(8/\epsilon_*)$,

$$|\mathscr{A}| = \pi\epsilon_*^2 l_*^2, \tag{6.14}$$

$$\kappa = U_* l_* \pi\{1+O(\delta_*^4)\}, \tag{6.15}$$

$$\tilde{W} = U_*\{[\tfrac{1}{4}\ln(8/\epsilon_*)-\tfrac{1}{16}]+\epsilon_*^2[-\tfrac{3}{32}\ln(8/\epsilon_*)+\tfrac{15}{128}]+O(\delta_*^4)\}, \tag{6.16}$$

$$P = \rho U_* l_*^3 \pi^2\{1+\tfrac{3}{4}\epsilon_*^2+O(\delta_*^4)\}, \tag{6.17}$$

$$\tilde{T} = \rho U_*^2 l_*^3 \pi^2\{[\tfrac{1}{2}\ln(8/\epsilon_*)-\tfrac{7}{8}]+\tfrac{3}{16}\epsilon_*^2\ln(8/\epsilon_*)+O(\delta_*^4)\}. \tag{6.18}$$

The result (6.16) is plotted in figure 4 up to rather large values of ϵ_* because of its success for Hill's spherical vortex. For this vortex $\mathscr{A}$ is a half disk of radius $2l_*$, so that $\epsilon_* = \surd 2$ by (6.14). The property $\omega/r = 15\tilde{W}/8l_*^2$ of Hill's vortex, and the definition (6.12c) of U_*, then imply the exact value $\tilde{W}/U_* = 4/15 \approx 0{\cdot}267$, while (6.16) gives the approximate value $0{\cdot}281$.

In terms of co-ordinates (S_*, T_*) defined by

$$r-l_* = \epsilon_* l_* S_* \cos T_*, \quad z = \epsilon_* l_* S_* \sin T_*,$$

the equation of $\partial\mathscr{A}$ is

$$S_* = 1-\epsilon_*^2[\tfrac{3}{8}\ln(8/\epsilon_*)-\tfrac{17}{32}]\cos 2T_* + \epsilon_*^3[\tfrac{21}{128}\ln(8/\epsilon_*)-\tfrac{273}{1024}] \times(\cos 3T_*-\cos T_*)+O(\delta_*^4), \tag{6.19}$$

and the position of the stagnation point in $\mathscr{A}$ is

$$\left.\begin{aligned} S_* &= (l-l_*)/\epsilon_* l_* = \tfrac{5}{8}\epsilon_* + \epsilon_*^3[-\tfrac{99}{128}\ln(8/\epsilon_*) + \tfrac{739}{1024}] + O(\delta_*^5), \\ T_* &= 0. \end{aligned}\right\} \qquad (6.20)$$

The dividing streamline $\psi = 0$ is also of interest because it separates the fluid permanently near the core from fluid streaming past. This streamline and the

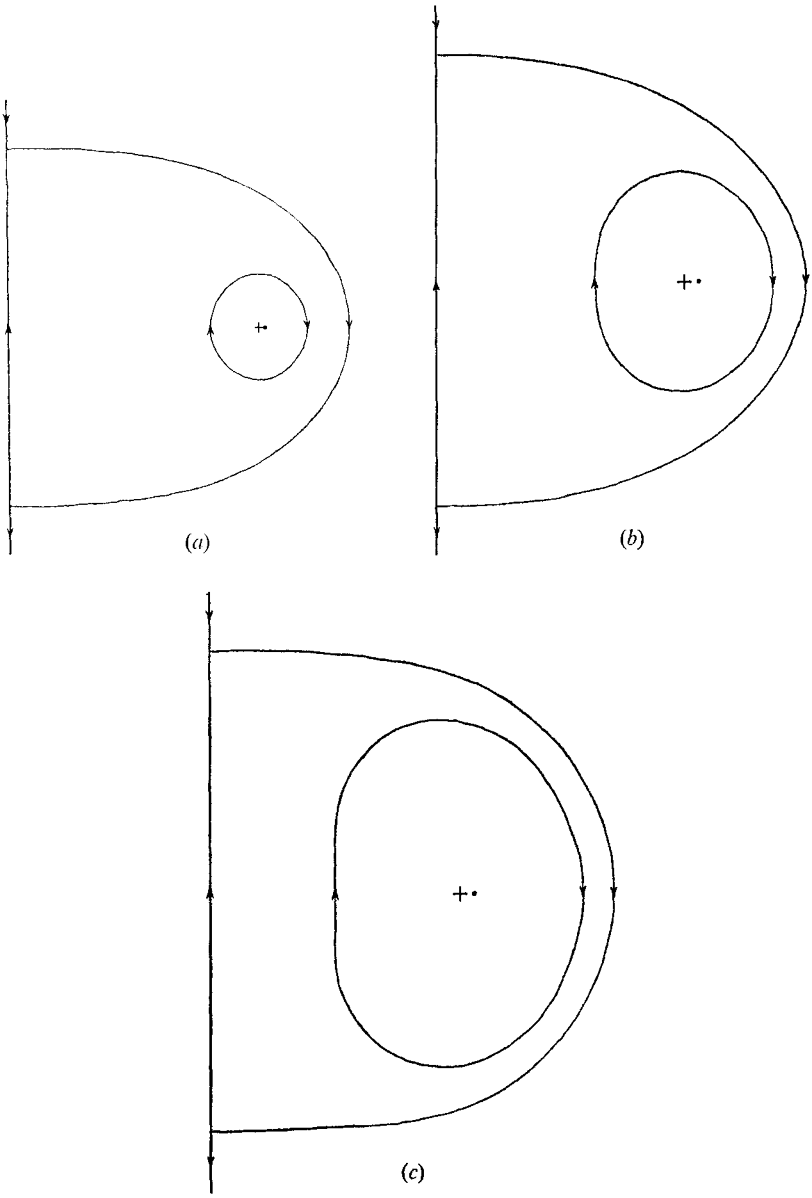


FIGURE 5. The core boundary $\partial\mathscr{A}$ and the dividing streamline $\psi = 0$ for ω/r = constant. The cross marks the mid chord point $r = l_*$, $z = 0$, and the dot the stagnatio n point $r = l$, $z = 0$. The parameter $\epsilon_* = 0{\cdot}2$, $0{\cdot}4$ and $0{\cdot}6$ for (*a*), (*b*) and (*c*) respectively.

core boundary $\partial\mathscr{A}$ are shown in figure 5 and again suggest that Hill's vortex (for which the two curves coincide) is approached as ϵ_* increases, even though the validity of our theory is assured only for sufficiently small values of ϵ_* (the exact upper bound being unknown). It may be that the basic step of replacing the kernel G by its expansion up to ϵ^3 is a good approximation for quite substantial values of ϵ, in view of the large domain of convergence of that series noted in (3.1 c); and our subsequent approximations have been merely consistent with that step.

The dividing streamlines $\psi = 0$ in figure 5 were computed by a combination of (a) the 'inner' expansion of the stream function used so far, and (b) the 'outer' expansion, which is that for $\epsilon \to 0$ with $(r-l)^2+z^2$ bounded away from zero. Both series are given in some detail by Fraenkel (1969) for the related problem of the magnetic field induced by a ring current.

There can be very little doubt that for uniform ω/r there exists a family of steady vortex rings characterized by the parameter ϵ_*, $0 < \epsilon_* \leqslant \sqrt{2}$, with Hill's vortex forming the largest member of the family. We know this assertion to be strictly true for some interval $0 < \epsilon_* < \alpha$; and J. Norbury, until recently a research student at Cambridge, has proved it (Norbury 1972 a) for some interval $\beta < \epsilon_* \leqslant \sqrt{2}$. Norbury's vortex rings are topologically equivalent to a torus for $\epsilon_* < \sqrt{2}$, but tend to Hill's spherical vortex as $\epsilon_* \to \sqrt{2}$. Norbury (1972 b) has also solved equation (2.1) numerically for intermediate values of ϵ_* (and for $F(\psi) = \text{constant}$).

For other vorticity distributions, we again know that families of steady vortex rings exist 'in the small'. It is not obvious what the largest member (if it exists) of such a family will be, but we may expect it to be an ovoid of some kind, that is, topologically equivalent to a sphere.

Appendix. Some computational details

The expansion of G up to ϵ^3

By virtue of its definition, the kernel $G(\mathbf{S}, \hat{\mathbf{S}}, \epsilon)$ satisfies the differential equation

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}-\frac{\epsilon}{1+\epsilon X}\frac{\partial}{\partial X}\right)G = 0 \tag{A 1}$$

for $\mathbf{S} \neq \hat{\mathbf{S}}$. The polynomials p_1 to P_3 in (3.1 b) were tested by checking that (a) the corresponding expansion of G satisfies (A 1) with an error $O(\epsilon^4 \ln \epsilon)$, and ($b$) the corresponding expansion of $(1+\epsilon\hat{X})^{-1}G$ is invariant when X and $\hat{X}$ are interchanged.

The forcing function g for $\Omega = 1$

With the notation $\lambda = \ln(8/\epsilon)$ and with $\Omega = 1$, the expansion up to ϵ^3 of the function g defined by (3.3 c) is given by the following:

$$\left.\begin{aligned} -g_1(\mathbf{s},\lambda) &= \tfrac{1}{4}\{(\lambda+1)\,s-\tfrac{5}{4}s^3\}\cos t,\\ -g_2(\mathbf{s},\lambda) &= \tfrac{1}{16}\{(\tfrac{3}{2}\lambda-\tfrac{3}{8})+(\lambda+1)\,s^2-\tfrac{7}{8}s^4\}\\ &\qquad +\tfrac{1}{16}\{(-\tfrac{1}{2}\lambda+\tfrac{21}{8})\,s^2-\tfrac{3}{4}s^4\}\cos 2t,\\ -g_3(\mathbf{s},\lambda) &= \tfrac{1}{128}\{-3(\lambda-1)\,s+(-3\lambda+\tfrac{13}{4})\,s^3+\tfrac{1}{4}s^5\}\cos t\\ &\qquad +\tfrac{1}{128}\{(\lambda-\tfrac{11}{4})\,s^3-\tfrac{1}{8}s^5\}\cos 3t.\end{aligned}\right\} \tag{A 2}$$

To interpret and to check this, we set $U = l = 1$ and note that according to (3.3a) the function

$$\Psi(S) + C_0 - g(\mathbf{S}, \epsilon) \equiv \Phi_0(\mathbf{S}, \epsilon), \quad \text{say}, \tag{A 3}$$

is the stream function due to vorticity $\omega/r = \Omega(S)/\epsilon^2$ in the torus $S < 1$ of exactly circular cross-section, the cross-sectional radius being ϵ. Accordingly

$$\left\{\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} - \frac{\epsilon}{1+\epsilon X}\frac{\partial}{\partial X}\right\}\Phi_0 = -(1+\epsilon X)^2\,\Omega(S) \quad (S < 1),$$

and this equation was used to check (A 2).

The nonlinear terms $\mathbf{M}_2(q_1, \lambda)$ *and* $\mathbf{M}_3(q_1, q_2, \lambda)$ *for* $\Omega = 1$

In this case the integral in (2.7) can be written

$$\frac{1}{2\pi}\left\{\iint_{\hat{S}<1} + \iint_{1<\hat{S}<1+q(1,\hat{T},\epsilon)} - \iint_{1+q(1,\hat{T},\epsilon)<\hat{S}<1} G(\mathbf{S}, \hat{\mathbf{S}}, \epsilon)\,\hat{S}\,d\hat{S}\,d\hat{T}\right\}$$
$$\equiv \Phi_0(\mathbf{S}, \epsilon) + \Phi_{\mathrm{I}}(\mathbf{S}, \epsilon), \quad \text{say},$$

where Φ_0 denotes the integral over $\hat{S} < 1$, as in (A 3). This leads to a way of computing the nonlinear terms $\mathbf{M}_2$ and $\mathbf{M}_3$ in equations (3.7c) and (3.7d) that is slightly more economical than the method given in § 3. (Both methods were used for $\mathbf{M}_2$.) The function $\Phi_{\mathrm{I}}(\mathbf{S}, \epsilon)$ is first calculated up to ϵ^3 in terms of the constants $\epsilon q_{1,1}(1)$, $\epsilon^2 q_{2,0}(1)$ and $\epsilon^2 q_{2,2}(1)$, which are the Fourier coefficients of ϵq_1 and $\epsilon^2 q_2$ on $s = 1$ (see (6.2)). To find the expansion up to ϵ^3 of the integral in (2.7), now to be written as a function of $\mathbf{s}$, we substitute

$$S = s + \epsilon q_1(\mathbf{s}, \lambda) + \epsilon^2 q_2(\mathbf{s}, \lambda), \quad T = t$$

into $\Phi_0 + \Phi_{\mathrm{I}}$, expand up to ϵ^3, and add $\epsilon^3(Vq_3 - \mathbf{L}q_3)$.

REFERENCES

Dyson, F. W. 1893 The potential of an anchor ring, I and II. *Phil. Trans. Roy. Soc.* A **184**, 43 and 1041.

Fraenkel, L. E. 1969 On the method of matched asymptotic expansions, III. *Proc. Camb. Phil. Soc.* **65**, 263.

Fraenkel, L. E. 1970 On steady vortex rings of small cross-section in an ideal fluid. *Proc. Roy. Soc.* A **316**, 29.

Lamb, H. 1932 *Hydrodynamics*. Cambridge University Press.

Maruhn, K. 1934 Über zwei Gleichgewichtsfiguren rotierender inhomogener Flüssigkeit. *Math. Zschr.* **39**, 244.

Maruhn, K. 1957 Über die Existenz stationärer Bewegungen von Wirbelringen. *Proc. Ninth International Congress Appl. Mech. Brussels*, **1**, 173.

Norbury, J. 1972a A steady vortex ring close to Hill's spherical vortex. To appear in *Proc. Camb. Phil. Soc.*

Norbury, J. 1972b A family of steady vortex rings. To be submitted to *J. Fluid Mech.*

Saffman, P. G. 1970 The velocity of viscous vortex rings. *Studies in Appl. Math.* **49**, 371.